Determining a Force Acting on a Plate – An Inverse Problem

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It is traditional in the study of elasticity to determine the response of a structure to a known force. Such problems may be described as direct problems as they involve the determination of the unknown effects of a known cause. The problem of determining the force acting on a structure from measurements of the response of the structure to the force is the inverse problem. Presented here is a method for determining the location and magnitude of a static point force acting on a simply-supported elastic rectangular plate from a number of displacement readings at discrete points on the plate. This problem reduces to a nonlinear least-squares one. It is solved by calculating an approximate solution from a simplified set of equations that is then used as an initial estimate in an iterative procedure for a solution of the actual nonlinear least-squares problem. Results of numerical simulations illustrate the use of the method, and a confidence criterion is supplied. Presented also is a demonstration of the robustness of the algorithm to the effects of measurement noise, as well as a means by which the method may be extended to problems of a more general nature.

Introduction

THREE distinct analytical problems[1] arise in the study of the response of systems:

1) Given the input and the system parameters, determine the output.
2) Given the system parameters and the output, determine the input.
3) Given the input and the output, determine the system parameters.

Problems of the first type are described as direct problems. Problems of the second type (the reconstruction problem) and the third type (the identification problem) are together described as inverse problems.

Inverse problems have received much attention. They include the inverse scattering problem and the seismological inverse kinematic problem. In the former, the shape of an object is deduced from a measured field of scattered electromagnetic radiation. In the latter, the propagation speed of artificially generated seismic waves is measured to determine the density and elastic constants of the medium through which the waves travel.

It is traditional in the study of elastic systems to determine the response (such as the displacement, acceleration, or strain) of a structure to a known force. Treated here is the inverse problem of force identification in which the force acting on a structure is determined from measurements of the response of the structure to the force. It is a relevant problem because it is not always feasible to measure the loading applied to a structure directly—the flow pressures imposed on an aircraft surface, for example.

Among force identification problems already treated is that of Ellis[2,3] who developed a method to determine the spatial distribution of the fluctuating aerodynamic forces on a flexible building model from strain and acceleration response measurements. Michaels and Pao[4] describe and demonstrate a method to determine experimentally the oblique force applied to the surface of an elastic plate. The parameters determined are the orientation and time history of the force. Chang and Sachse[5] consider the inverse problem of an extended, finite source of elastic waves in a thick plate. Determined are the time dependence and spatial distribution of the source. The source is treated as a superposition of point sources, each of known location. Stephens[6] provides an extensive overview of force identification problems.

Presented here is a method for determining the location and magnitude of a static point force acting on a simply-supported elastic rectangular plate from a number of displacement readings at discrete points on the plate. It turns out that the matter reduces to a nonlinear least-squares problem. Its solution rests on a preliminary calculation of an approximate analytical solution that is used as an initial estimate for an iterative procedure leading to the solution of the actual nonlinear least-squares problem. Presented also are results of numerical simulations that illustrate the use of the method and demonstrate the level of confidence that can be placed on the result. Also demonstrated is the robustness of the solution algorithm to the effects of measurement noise, and a means by which the method may be extended to solve problems involving more general structural configurations and loadings is shown. In this, the work reported here represents a step taken toward the solution of practical force identification problems.

Force-Displacement Relation

The deflection at a point $Q(x,y)$ due to a static point load P at the point (ξ,η) of an elastic rectangular plate simply supported along all of its edges (Fig. 1) is given[7] by

$$w = \frac{4P}{\pi^4 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{\sin(m\pi\xi/a)\sin(n\pi\eta/b)\sin(m\pi x/a)\sin(n\pi y/b)}{[(m^2/a^2)+(n^2/b^2)]^2} \tag{1}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

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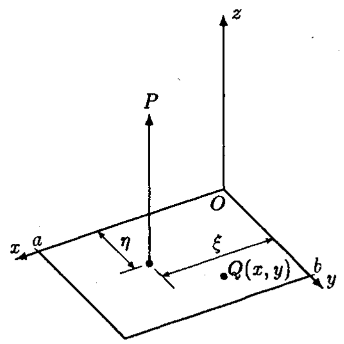

Fig. 1 Problem geometry.

and where D is the flexural rigidity, E the modulus of elasticity, h the plate thickness, and ν the Poisson's ratio. The inverse problem of determining the static point force from displacement measurements is then to determine the location (ξ,η) and magnitude P of the force from a knowledge of the deflection, w_i, $i = 1,2,\ldots,N$, at the N points, $Q_i(x_i,y_i)$, $i = 1,2,\ldots,N$, on the plate. In practice, these deflections may be determined by the placement of an array of transducers.

Approximate Solution

Equation (1) may be written as

$$W_i = \frac{w_i}{a} = A\,[A_{11}^{(i)} \sin\alpha \sin\beta + A_{12}^{(i)} \sin\alpha \sin 2\beta + A_{21}^{(i)} \sin 2\alpha \sin\beta + H^{(i)}] \tag{2}$$

where (a, b, and D being given)

$$A = (4/\pi^4)(a/b)(Pa/D)$$

is the force parameter (which may, without loss of generality, be treated as the force magnitude),

$$\alpha = \pi\xi/a \qquad \beta = \pi\eta/b$$

are the nondimensional coordinates of P,

$$A_{mn}^{(i)} = \frac{\sin(m\pi x_i/a)\,\sin(n\pi y_i/b)}{[m^2 + (a/b)^2 n^2]^2}$$

are the modal coefficients of the displacement response of the ith transducer, and

$$H^{(i)} = \sum_{m=3}^{\infty} A_{m1}^{(i)} \sin m\alpha \sin\beta + \sum_{n=3}^{\infty} A_{1n}^{(i)} \sin\alpha \sin n\beta + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} A_{mn}^{(i)} \sin m\alpha \sin n\beta$$

represents the higher-order terms. A second transducer reading, w_j, generates another nondimensional equation of the same form as that of Eq. (2). Divide one by the other to gain

$$[A_{11}^{(j)} - \lambda_{ij}A_{11}^{(i)}] \sin\alpha \sin\beta + [A_{12}^{(j)} - \lambda_{ij}A_{12}^{(i)}] \sin\alpha \sin 2\beta + [A_{21}^{(j)} - \lambda_{ij}A_{21}^{(i)}] \sin 2\alpha \sin\beta + H^{(j)} - \lambda_{ij}H^{(i)} = 0 \tag{3}$$

where $\lambda_{ij} = W_j/W_i = w_j/w_i$. The effect of this is to remove the force parameter A and hence the force magnitude P from consideration. Now divide throughout by $\sin\alpha \sin\beta$ to give

$$[A_{11}^{(j)} - \lambda_{ij}A_{11}^{(i)}] + 2[A_{12}^{(j)} - \lambda_{ij}A_{12}^{(i)}] \cos\beta + 2[A_{21}^{(j)} - \lambda_{ij}A_{21}^{(i)}] \cos\alpha + \frac{H^{(j)} - \lambda_{ij}H^{(i)}}{\sin\alpha \sin\beta} = 0 \tag{4}$$

where

$$\frac{H}{\sin\alpha \sin\beta} = \sum_{m=3}^{\infty} A_{m1} \frac{\sin m\alpha}{\sin\alpha} + \sum_{n=3}^{\infty} A_{1n} \frac{\sin n\beta}{\sin\beta} + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} A_{mn} \frac{\sin m\alpha \sin n\beta}{\sin\alpha \sin\beta}$$

Because

$$\frac{\sin(2n+1)\theta}{\sin\theta} = 2[\cos 2n\theta + \cos(2n-2)\theta + \ldots + \cos 2\theta + ½] \qquad n = 1,2,\ldots \tag{5}$$

$$\frac{\sin 2n\theta}{\sin\theta} = 2[\cos(2n-1)\theta + \cos(2n-3)\theta + \ldots + \cos 3\theta + \cos\theta] \qquad n = 1,2,\ldots \tag{6}$$

Equation (4) can be succinctly recast as

$$D_{ij} + E_{ij}\cos\beta + F_{ij}\cos\alpha = \epsilon_{ij} \tag{7}$$

where

$$D_{ij} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} [A_{2m+1\,2n+1}^{(j)} - \lambda_{ij}A_{2m+1\,2n+1}^{(i)}]$$

$$E_{ij} = 2\sum_{m=0}^{\infty}\sum_{n=½}^{\infty} [A_{2m+1\,2n+1}^{(j)} - \lambda_{ij}A_{2m+1\,2n+1}^{(i)}]$$

$$F_{ij} = 2\sum_{m=½}^{\infty}\sum_{n=0}^{\infty} [A_{2m+1\,2n+1}^{(j)} - \lambda_{ij}A_{2m+1\,2n+1}^{(i)}]$$

$$\epsilon_{ij} = -K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=p/2}^{\infty}\sum_{n=q/2}^{\infty} [A_{2m+1\,2n+1}^{(j)} - \lambda_{ij}A_{2m+1\,2n+1}^{(i)}] \cos p\alpha \cos q\beta$$

with $K = 0$ when $p = q = 0$, $p = 1$ and $q = 0$ or $p = 0$ and $q = 1$, $K = 2$ when $p = 0$ and $q > 1$ or $p > 1$ and $q = 0$, and $K = 4$ when $p \neq 0$ and $q \neq 0$. Note that in the preceding summations all indices increment, necessarily, by unity.

It may be seen that the coefficients on the right-hand side of Eq. (7) contain only the higher-order terms in A_{mn}. For N transducer readings, there are $N - 1$ equations of this form. The approximate solution relies, for its accuracy, on the coefficients on the left-hand side dominating those on the right-hand side. Then, the terms on the right-hand side that, in effect, arise from a series truncation, may be treated as "noise" and hence as random variables. In so doing, the effect of the higher-order terms in $\cos p\alpha \cos q\beta$, $p = 0,1,\ldots$ and $q = 0,1,\ldots$ may be isolated, leaving a simpler set of equations to be solved.

If two random variables, x and y, have the probability density function $f(x,y)$, and $\phi(x,y)$ is an arbitrary function of x and y, the expected value of $\phi(x,y)$ is given[8] by

$$E[\phi(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y) f(x,y)\,dx\,dy \tag{8}$$

Put ϵ_{ij} in the form

$$\epsilon_{ij} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} C_{pq}^{(ij)} \cos p\alpha \cos q\beta, \; i \text{ fixed}, \; j = 1,2,\ldots,N \text{ but } \neq i \tag{9}$$

where

$$C_{pq}^{(ij)} = -K\sum_{m=p/2}^{\infty}\sum_{n=q/2}^{\infty} [A_{2m+1\,2n+1}^{(j)} - \lambda_{ij}A_{2m+1\,2n+1}^{(i)}]$$

With α and β both being uniformly distributed over the interval $[0,\pi]$, $E(\epsilon_{ij})$ and $E(\epsilon_{ij}\epsilon_{ik})$, respectively, are given by

$$E(\epsilon_{ij}) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} C_{pq}^{(ij)} \cos p\alpha \cos q\beta \, d\alpha \, d\beta$$

$$= 0 \tag{10}$$

$$E(\epsilon_{ij}\epsilon_{ik}) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \left\{\left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} C_{pq}^{(ij)} \cos p\alpha \cos q\beta\right] \times \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} C_{pq}^{(ik)} \cos p\alpha \cos q\beta\right]\right\} d\alpha \, d\beta$$

$$= \frac{1}{2}\sum_{p=0}^{\infty} C_{p0}^{(ij)}C_{p0}^{(ik)} + \frac{1}{2}\sum_{q=0}^{\infty} C_{0q}^{(ij)}C_{0q}^{(ik)} + \frac{1}{4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} C_{pq}^{(ij)}C_{pq}^{(ik)} \tag{11}$$

Having determined $E(\epsilon_{ij})$ and $E(\epsilon_{ij}\epsilon_{ik})$, it is then possible to formulate the variance-covariance matrix of the random variables on the right-hand side of the $N-1$ equations. As the covariance σ_{xy} of two random variables x and y is given by

$$\sigma_{xy} = E(xy) - E(x)E(y) \tag{12}$$

it follows that term jk of this variance-covariance matrix is given by

$$V_{jk} = \frac{1}{2}\sum_{p=0}^{\infty} C_{p0}^{(ij)}C_{p0}^{(ik)} + \frac{1}{2}\sum_{q=0}^{\infty} C_{0q}^{(ij)}C_{0q}^{(ik)} + \frac{1}{4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} C_{pq}^{(ij)}C_{pq}^{(ik)} \tag{13}$$

The solution of the $N-1$ equations is then obtained by minimizing the scalar $\epsilon^T V^{-1}\epsilon$, where ϵ is the vector whose $N-1$ terms are given by Eq. (9) and V is the variance-covariance matrix. This leads to the set of equations

$$\begin{bmatrix} F^TV^{-1}F & F^TV^{-1}E \\ F^TV^{-1}E & E^TV^{-1}E \end{bmatrix}\begin{bmatrix}\cos\alpha \\ \cos\beta\end{bmatrix} = -\begin{bmatrix} D^TV^{-1}F \\ D^TV^{-1}E\end{bmatrix} \tag{14}$$

where D, E, and F are vectors whose $N-1$ terms are given, respectively, by D_{ij}, E_{ij}, and F_{ij} with i fixed and $j = 1,2,\ldots,N$, but $\neq i$. Solve for $\cos\alpha$ and $\cos\beta$ and obtain, as an approximate solution for the force location,

$$\xi_e = (a/\pi)\cos^{-1}(\hat{\cos}\alpha) \quad \text{and} \quad \eta_e = (b/\pi)\cos^{-1}(\hat{\cos}\beta) \tag{15}$$

where $\hat{\cos}\alpha$ and $\hat{\cos}\beta$ are the values obtained from Eq. (14) for $\cos\alpha$ and $\cos\beta$, respectively. However, if the solutions for the magnitudes of $\hat{\cos}\alpha$ and/or $\hat{\cos}\beta$ should prove to be greater than unity, the force is taken to be close to the edge of the plate. It is then appropriate to adopt the following strategy:

$$\begin{aligned} &\text{If } \hat{\cos}\alpha < -1, \ \xi_e = (1-e)a \\ &\text{If } \hat{\cos}\alpha > 1, \ \xi_e = ea \\ &\text{If } \hat{\cos}\beta < -1, \ \eta_e = (1-e)b \\ &\text{If } \hat{\cos}\beta > 1, \ \eta_e = eb \end{aligned} \tag{16}$$

where $0 < e \ll 1$.

From the approximate solution, (ξ_e, η_e), for the force location, an estimate for the force parameter is determined by implementing the condition $(\partial S/\partial A_e) = 0$ to the least-squares objective function $S = \sum_{i=1}^{N}(W_{i_e} - W_{i_o})^2$, where W_{i_e} is the estimated displacement of transducer i and W_{i_o} its observed displacement, both nondimensionalized by the plate length a. This leads to an expression for the force parameter:

$$A_e = \frac{\sum_{i=1}^{N} W_{i_o}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{\sin(m\pi\xi_e/a)\sin(n\pi\eta_e/b)\sin(m\pi x_i/a)\sin(n\pi y_i/b)}{[m^2+(a/b)^2n^2]^2}\right]}{\sum_{i=1}^{N}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{\sin(m\pi\xi_e/a)\sin(n\pi\eta_e/b)\sin(m\pi x_i/a)\sin(n\pi y_i/b)}{[m^2+(a/b)^2n^2]^2}\right]^2} \tag{17}$$

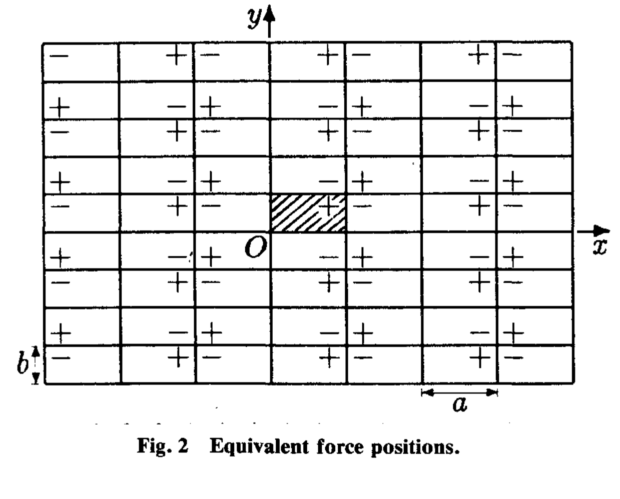

Fig. 2 Equivalent force positions.

Iterative Procedure

A more accurate solution may now be obtained by minimizing the least-squares nonlinear objective function $S = \sum_{i=1}^{N}(W_{i_e} - W_{i_o})^2$, using the approximate solution as an initial estimate.

If $x^{(k)} = [x_1^{(k)}x_2^{(k)}x_3^{(k)}]^T$ is the kth estimate of the solution vector $[\xi\eta A]^T$ and $\delta^{(k)}$ is the change in the kth estimate after one iteration, then

$$x^{(k+1)} = x^{(k)} + \delta^{(k)} \tag{18}$$

In the Gauss-Newton method,[9] $\delta^{(k)}$ is determined from the solution of the equation

$$G^{(k)}\delta^{(k)} = -g^{(k)} \tag{19}$$

where

$$G_{jk} = 2\sum_{i=1}^{N}\frac{\partial W_{i_e}}{\partial x_j}\frac{\partial W_{i_e}}{\partial x_k}, \qquad j = 1,2,3 \quad \text{and} \quad k = 1,2,3$$

$$g_j = 2\sum_{i=1}^{N}(W_{i_e} - W_{i_o})\frac{\partial W_{i_e}}{\partial x_j}, \qquad j = 1,2,3$$

the derivatives $\partial W_{i_e}/\partial x_j$, $j = 1,2,3$ being determined analytically.

The Gauss-Newton method is applicable to least-squares problems in general. It is, however, advantageous to make some modifications that make use of the structure of the problem. The first is to update the estimate of the force magnitude at each step to the value predicted by Eq. (17). By this it is meant that at each iterative step the estimate of the force location is obtained from Eq. (18) while the estimate of the force magnitude is obtained by using this estimate of the force location in Eq. (17).

The iterative procedure described so far must be constrained so that the force lies on the plate, that is, so that

$$0<\xi_e<a \quad \text{and} \quad 0<\eta_e<b \tag{20}$$

As

$$\sin k\theta = \sin k(\theta + 2l\pi)$$

$$\sin k\theta = -\sin k(2\pi - \theta + 2l\pi) \tag{21}$$

for $k = 1,2,\ldots$ and $l = \ldots, -1,0,1,\ldots$, the force may act at any one of the positions shown in Fig. 2, in which the plus sign means that a force of $+P$ is acting while the minus sign means that a force of $-P$ is acting, and still produce an identical displacement field. Figure 2 also shows the actual plate (shaded) and all its immediately adjacent plates. A scheme can be formulated in which the results of an iterative step may be returned from any of the adjacent plates to the actual plate. This scheme reads as follows:

$$\text{If } a<\xi_e<2a \quad \text{and} \quad 0<\eta_e<b$$

$$\text{or } 0<\xi_e<a \quad \text{and} \quad b<\eta_e<2b$$

$$\text{or } -a<\xi_e<0 \quad \text{and} \quad 0<\eta_e<b$$

$$\text{or } 0<\xi_e<a \quad \text{and} \quad -b<\eta_e<0, \text{ then set } P \text{ to } -P$$

$$\text{Then if } \xi_e<0, \text{ set } \xi_e \text{ to } -\xi_e, \text{ if } \eta_e<0, \text{ set } \eta_e \text{ to } -\eta_e$$

$$\text{if } \xi_e>a, \text{ set } \xi_e \text{ to } 2a-\xi_e, \text{ if } \eta_e>b, \text{ set } \eta_e \text{ to } 2b-\eta_e \tag{22}$$

This is akin to reflecting the force in the plate boundaries until it locates on the plate. This reflection is justified on the grounds that the Gauss-Newton method does not distinguish between the actual force and any of its images.

Now, it is also possible for the results of an iterative step to lie outside the region defined by the plate and its eight immediately adjacent plates. When this occurs, the Gauss-Newton method is bypassing the force and all of its closest images. In such circumstances, it would seem more appropriate to halve the step size predicted by the Gauss-Newton method and to recalculate the force magnitude from Eq. (17) repeatedly until the force lies within the nine-part region defined by the plate and its immediately adjacent plates. Only then should the scheme, Eq. (22), be implemented.

Uniqueness

It is possible to demonstrate, via numerical simulation, that three transducer readings may lead to a finite number of stable, nonunique solutions. By stable, it is meant that there exists a finite range of movement of any or all of the transducers that does not affect the existence of these nonunique solutions. The added information provided by a fourth transducer might reasonably be expected to ensure a unique solution. The reason for this is as follows. With two transducer readings, it may be shown that the coordinates of the estimated location of the force (ξ_e,η_e) must satisfy the equation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\pi x_j/a)\sin(n\pi y_j/b)-\lambda_{ji}\sin(m\pi x_i/a)\sin(n\pi y_i/b)}{[(m^2/a^2)+(n^2/b^2)]^2}\sin\frac{m\pi\xi_e}{a}\sin\frac{n\pi\eta_e}{b}=0 \tag{23}$$

where

$$\lambda_{ji}=\frac{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{\sin(m\pi\xi/a)\sin(n\pi\eta/b)\sin(m\pi x_j/a)\sin(n\pi y_j/b)}{[(m^2/a^2)+(n^2/b^2)]^2}}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{\sin(m\pi\xi/a)\sin(n\pi\eta/b)\sin(m\pi x_i/a)\sin(n\pi y_i/b)}{[(m^2/a^2)+(n^2/b^2)]^2}}$$

Table 1 Results of least-squares calculation (initial estimate and iterates)

Force coordinates ξ_e	Force coordinates η_e	Force magnitude A_e	Objective function
0.598100	0.0967481	0.249920	0.208230×10^{-8}
0.419772	0.132189	0.198574	0.186367×10^{-7}
0.764832	0.0288785	0.116417	0.460225×10^{-8}
0.564032	0.0766440	0.402787	0.980881×10^{-9}
0.517361	0.109187	0.683709	0.806857×10^{-9}
0.532269	0.132503	0.788123	0.635018×10^{-11}
0.533880	0.143106	0.951763	0.957107×10^{-13}
0.533668	0.144596	0.988206	0.560469×10^{-14}
0.533616	0.144926	0.996785	0.324478×10^{-15}
0.533604	0.145013	0.999044	0.230678×10^{-16}
0.533601	0.145038	0.999700	0.197591×10^{-17}
0.533600	0.145045	0.999903	0.191512×10^{-18}
0.533600	0.145048	0.999968	0.199037×10^{-19}
0.533599	0.145049	0.999989	0.214467×10^{-20}
0.533599	0.145049	0.999996	0.235257×10^{-21}
0.533599	0.145049	0.999999	0.260298×10^{-22}
0.533599	0.145049	1.00000	0.289192×10^{-23}
0.533599	0.145049	1.00000	0.321921×10^{-24}
0.533599	0.145049	1.00000	0.358683×10^{-25}
0.533599	0.145049	1.00000	0.399821×10^{-26}
0.533599	0.145049	1.00000	0.445763×10^{-27}

Dimensions of plate: $a = 1$, $b = 0.179902$
Actual force coordinates: $\xi = 0.533599$, $\eta = 0.145049$
Actual force magnitude: $A = 1$

Making use of the relation $\sin z = (e^{iz} - e^{-iz})/2i$, Eq. (23) may be written as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn_o}^{(ij)}\left(z_{1_e}^m-\frac{1}{z_{1_e}^m}\right)\sin\frac{n\pi\eta_e}{b}=0 \tag{24}$$

where

$$A_{mn_o}^{(ij)}=\frac{\sin(m\pi x_j/a)\sin(n\pi y_j/b)-\lambda_{ji}\sin(m\pi x_i/a)\sin(n\pi y_i/b)}{[(m^2/a^2)+(n^2/b^2)]^2}$$

$$z_{1_e}=e^{i(\pi\xi_e/a)}$$

With a third transducer at, say (x_k,y_k), the coordinates of the estimated location must also satisfy the equation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn_o}^{(ik)}\left(z_{1_e}^m-\frac{1}{z_{1_e}^m}\right)\sin\frac{n\pi\eta_e}{b}=0 \tag{25}$$

For sufficiently large values of m and n, the terms of the infinite series of Eqs. (24) and (25) become negligible. This then results in the two polynomials

$$a_0z_{1_e}^p+a_1z_{1_e}^{p-1}+\cdots+a_{p-1}z_{1_e}+a_p=0$$

$$b_0z_{1_e}^q+b_1z_{1_e}^{q-1}+\cdots+b_{q-1}z_{1_e}+b_q=0 \tag{26}$$

where p and q are finite, and $a_0, a_1,\ldots,a_p$ and $b_0, b_1,\ldots,b_q$ are polynomials of finite degree in $z_{2_e}=e^{i(\pi\eta_e/b)}$. A solution is obtained if the two polynomials have a root in common. If $q \leq p$, the necessary and sufficient condition for this to occur is[10]

$$\det S = 0 \tag{27}$$

where

$$S = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_p & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{p-1} & a_p & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{p-1} & a_p \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & b_q & 0 & \cdots & 0 & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & b_{q-1} & b_q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 & \cdots & \cdots & \cdots & \cdots & \cdots & b_{q-1} & b_q \end{bmatrix}$$

Before proceeding, two points must be noted. First, the polynomials of Eq. (26) should be deflated to eliminate any trivial roots in common, and second, z_{1_e} (and z_{2_e}) must, by definition, be of unit modulus.

Equation (27) leads to a polynomial in z_{2_e} of finite degree and will therefore have a finite number of solutions for z_{2_e} and hence η_e. For each of these values of η_e, Eq. (26) can only have a finite number of roots in common, resulting therefore in a finite number of values for ξ_e. Hence, with three transducer readings, there are a finite number of solution pairs (ξ_e, η_e).

A fourth transducer reading will lead to the third polynomial of the form of those of Eq. (26). It will have the root in common with them that corresponds to the true solution. However, should the polynomials of Eq. (26) have more than one root in common, there is no reason that this third polynomial should generically share with them a root other than the one corresponding to the true solution. By this it is meant that if the transducers are configured such that the three polynomials have more than one root in common, an arbitrary perturbation of any one transducer from its position will eliminate all but the root corresponding to the true solution. It is thus concluded that, generically, four transducers are required for a unique solution.

Results of Numerical Simulations

The method described previously for the simply-supported rectangular plate of Fig. 1 was implemented on a digital computer. The plate dimensions were denoted by a and b, the force parameter by A, and the force coordinates by ξ and η. Without loss of generality, the value of unity was assigned to each of a and A, so that b was then the plate aspect ratio. A total of 50 test cases was formed by generating b, ξ, and η randomly.

The results of a sample case appear in Table 1. Four transducers configured as illustrated in Fig. 3 were used to achieve these results. It is possible to attach a degree of confidence to them without making use of any a priori knowledge of the force location or magnitude. If $\boldsymbol{H}$ is the inverse of matrix $\boldsymbol{G}$ of Eq. (19), then an unbiased estimate of the covariance of the ith and jth parameters is[11]

$$\sigma_{x_i x_j} = \frac{2S}{m-n} H_{ij} \tag{28}$$

where S is the least-squares objective function, m is the number of transducer readings, and n is the number of parameters estimated. Equation (28) may therefore be used to evaluate the symmetric variance-covariance matrix of the solution vector $[x_1\ x_2\ x_3]^T = [\xi_e\ \eta_e\ A_e]^T$. For the sample case of Table 1 this turns out to be

$$0.213829 \times 10^{-21} \begin{bmatrix} 1 & -2.30487 & -68.0983 \\ -2.30487 & 595.396 & 14980.7 \\ -68.0983 & 14980.7 & 377118 \end{bmatrix}$$

As may be seen, the estimate for ξ has a smaller variance than the estimate for η. This is because the force is much closer to the boundary $y = b$ than it is to either of the boundaries $x = 0$ or $x = a$. On the whole, however, the magnitude of the preceding entries allows a great deal of certainty to be placed on the solution vector.

Finally, it should be mentioned that three other cases were not analyzed as the aspect ratios of the plates involved (9.5×10^{-4}, 9.7×10^{-5}, and 3.4×10^{-3}) were considered too small to be treated as those of a plate. In all remaining cases, the problem was solved successfully. In four of these cases, additional transducers were required. This was attributed to the flatness of the objective function near the solution. The reason for this may be seen from Fig. 4 that shows the displacement field (for a force magnitude of -1) for a case that required additional transducers for successful solution. In common with all other such cases, the force is located near the corner of a plate with a small aspect ratio. When this occurs, it may be seen that transducers away from the force provide relatively little information about the force, thereby necessitating the use of more transducers to provide the additional information required.

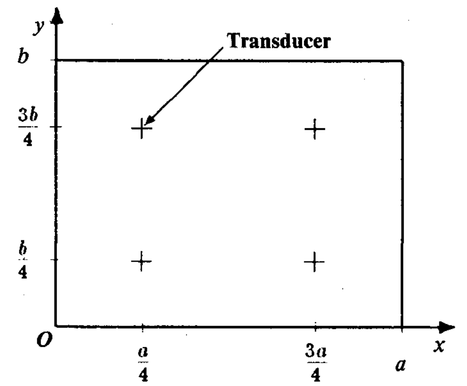

Fig. 3 Transducer configuration with four transducers.

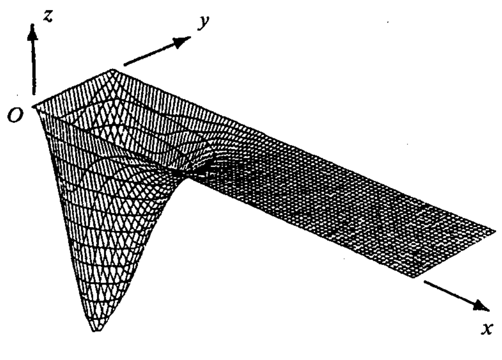

Fig. 4 Displacement field requiring additional transducers.

Effect of Measurement Noise

The previous solution technique was demonstrated using a number of displacement readings generated numerically without the effect of measurement noise being simulated. As the problem can be analyzed as a nonlinear least-squares problem, and as least-squares solution procedures are robust to the effects of measurement noise, it is to be expected that the addition of measurement noise will not have a significant effect on the results.

To demonstrate this point, the 50 test cases were run with Gaussian noise added (with a standard deviation set to 1% of the rms of the transducer readings). As before, the three cases with aspect ratios too small to be those of a plate were not analyzed. Of the remaining 47 cases, 38 were successfully solved with four transducer readings, four with five transducer readings, and the same number with six transducer readings. One case required nine transducer readings before it was successfully solved. The reason that this case required such a large number of transducer readings is that the force is located in the corner of a plate with a small aspect ratio, thereby leading, as previously explained, to a relatively flat objective function. (In the absence of measurement noise, this case required six transducer readings for its successful solution.) The results of the same case as that of Table 1 appear in Table 2, for the same four-transducer configuration.

It may therefore be concluded that the solution method is robust to the effects of measurement noise, which would be present in practice. The greater number of transducers required for the successful solution of some cases compared to the number of transducers required in the absence of noise is expected because of the presence of the measurement noise.

Extension to General Structures and Loadings

The method outlined preceding is applicable to a simply-supported rectangular plate acted upon by a single point force. It may be extended to tackle more general structures and loadings in the following manner.

First, consider the extension to more general structures. This may be made possible, for example, through a finite-element formulation as follows. Define an objective function by

$$S = \sum_{i=1}^{N} (w_{i_e} - w_{i_o})^2 \tag{29}$$

where w_{i_e} is the displacement of transducer i determined by the finite-element model for the force acting at (ξ_e,η_e), w_{i_o} is the measured or observed displacement of transducer i, and N is the number of transducer readings. If it is assumed that the displacement function is linear in force magnitude A, then there may be written

$$w_{i_e} = A_e f(\xi_e,\eta_e,x_i,y_i) \tag{30}$$

where A_e is the estimated force magnitude and (x_i,y_i) are the coordinates of transducer i [cf. Eq. (1)]. Using Eqs. (29) and (30) in conjunction with the condition $(\partial S/\partial A_e) = 0$ leads to the expression

$$A_e = \frac{\sum_{i=1}^{N} w_{i_o} f(\xi_e,\eta_e,x_i,y_i)}{\sum_{i=1}^{N}[f(\xi_e,\eta_e,x_i,y_i)]^2} \tag{31}$$

for the estimated magnitude of the force [cf. Eq. (17)]. Now the function $f(\xi_e,\eta_e,x_i,y_i)$ is the displacement at (x_i,y_i) due to a force of unit magnitude at (ξ_e,η_e) and may therefore be evaluated numerically by the finite-element model. The objective function defined in Eq. (29) may therefore be evaluated for the force acting at various points on the structure. By doing this over a coarse grid, and initial estimate for the force location and magnitude may be obtained by choosing the location that results in the smallest objective function.

Having obtained an initial estimate, the iterative scheme outlined previously may be used to improve upon it in the following manner. It may be seen from Eqs. (18) and (19) that each iterative step requires the evaluation of first-order derivatives of the displacement function of Eq. (30). These may be evaluated numerically by using the results of the finite-element model for $f(\xi_e,\eta_e,x_i,y_i)$ at and near the current estimate (ξ_e,η_e) of the force location. Knowing these derivatives thereby enables the matrix $\boldsymbol{G}$ and vector $\boldsymbol{g}$ to be computed so that the iterative scheme may be implemented.

Now, consider the extension to more general loadings. If, for instance, the plate is acted upon by a number of point loads, $P_j, j = 1,2,\ldots,M,$ the displacement field, by linear superposition, is given by[12]

$$w(x,y) = \sum_{j=1}^{M} \frac{4P_j}{\pi^4 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{\sin(m\pi\xi_j/a)\sin(n\pi\eta_j/b)\sin(m\pi x/a)\sin(n\pi y/b)}{[(m^2/a^2)+(n^2/b^2)]^2} \tag{32}$$

Table 2 Results of least-squares calculation with noise (initial estimate and iterates)

Force coordinates ξ_e	η_e	Force magnitude A_e	Objective function
0.597074	0.0979741	0.254669	0.531718×10^{-9}
0.422609	0.133711	0.217466	0.463308×10^{-8}
0.731257	0.0262643	0.129409	0.117047×10^{-8}
0.608066	0.139695	0.336925	0.597084×10^{-9}
0.476959	0.107222	0.498995	0.255578×10^{-8}
0.551847	0.136899	0.678584	0.120780×10^{-9}
0.528960	0.124947	0.733806	0.133963×10^{-10}
0.533580	0.152875	1.26016	0.378483×10^{-12}
0.532830	0.152300	1.24537	0.112218×10^{-12}
0.532852	0.150991	1.19370	0.113769×10^{-12}
0.532915	0.149442	1.13812	0.113879×10^{-12}
0.532989	0.147751	1.08383	0.114797×10^{-12}
0.533067	0.146121	1.03689	0.116440×10^{-12}
0.533137	0.144837	1.00311	0.118198×10^{-12}
0.533189	0.144081	0.984336	0.119428×10^{-12}
0.533217	0.143770	0.976779	0.119978×10^{-12}
0.533229	0.143681	0.974614	0.120130×10^{-12}
0.533232	0.143663	0.974177	0.120150×10^{-12}
0.533232	0.143661	0.974125	0.120147×10^{-12}
0.533232	0.143661	0.974128	0.120145×10^{-12}
0.533232	0.143661	0.974132	0.120144×10^{-12}
0.533232	0.143661	0.974133	0.120144×10^{-12}
0.533232	0.143661	0.974134	0.120144×10^{-12}
0.533232	0.143662	0.974134	0.120144×10^{-12}

Dimensions of plate: $a = 1$, $b = 0.179902$
Actual force coordinates: $\xi = 0.533599$, $\eta = 0.145049$
Actual force magnitude: $A = 1$
Level of noise: s.d. = 1% of rms

Or consider the case when the static load is replaced by the harmonic load $P\cos(\lambda t+\phi)$. The displacement field is then given by[12]

$$w(x,y,t)=\frac{4P}{\bar{\rho}ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\times\left\{\frac{\sin(m\pi\xi/a)\sin(n\pi\eta/b)\sin(m\pi x/a)\sin(n\pi y/b)}{(\lambda^2-p_{mn}^2)^2+p_{mn}^4\tau_K^2\lambda^2}\times[p_{mn}^2\tau_K\lambda\sin(\lambda t+\phi)-(\lambda^2-p_{mn}^2)\cos(\lambda t+\phi)]\right\} \quad (33)$$

where $\bar{\rho}$ is the surface density, $p_{mn}^2=(D\pi^4/\bar{\rho})\ [(m^2/a^2)+(n^2/b^2)]^2$, and τ_K is the retardation time. Because of the similarity of these displacement functions to that of Eq. (1), the method described for the single static point load may be used as a basis to tackle these more difficult inverse problems.

Discussion and Conclusion

Presented previously is a method for determining the location and magnitude of a static point force acting on a simply-supported elastic rectangular plate from a number of displacement readings at discrete points on the plate. In practice, these displacement readings would be subject to measurement noise. This is modeled by Gaussian noise, and it is shown that the solution algorithm is robust to its effect. This is to be expected because of the robustness of least-squares solution procedures to the effect of measurement noise.

The method is demonstrated by its application to the problem of a point load acting on a rectangular plate. Presented also is a means by which it may be extended to solve a larger class of problems. The extension provides both a means by which structures of a more arbitrary nature and loadings of a more general nature may be treated.

Finally, it may be seen that because of the good results obtained in the test cases considered, the method may be used with some confidence. It may therefore be concluded that the method outlined for the determination of the location and magnitude of a static point force acting on a simply-supported elastic rectangular plate, apart from being accurate and robust to the effects of measurement noise, may be extended to solve problems of a more general nature. In this, it represents a step taken toward the solution of practical force identification problems.

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